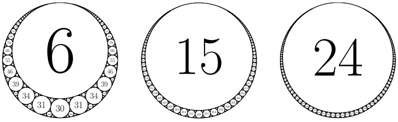
The Local-Global Conjecture for Apollonian Circle Packings

Summer Haag, **Clyde Kertzer**, James Rickards,
Katherine E. Stange

University of Colorado Boulder



(June 15, 2024)

Descartes Quadruples

Descartes Quadruples

Definition

Descartes Quadruples

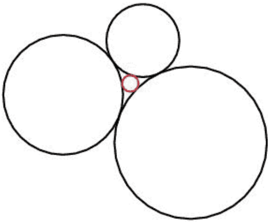
Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.

Descartes Quadruples

Definition

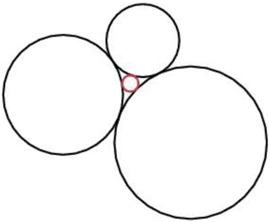
Descartes quadruple: four mutually tangent circles with disjoint interiors.

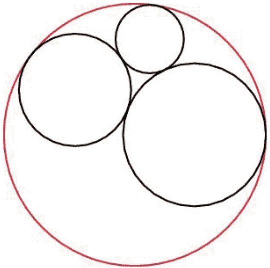


Descartes Quadruples

Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.

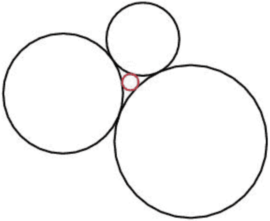


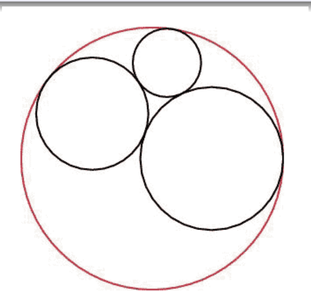


Descartes Quadruples

Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.



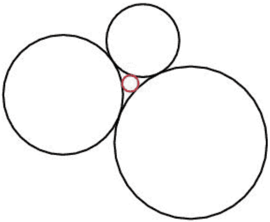


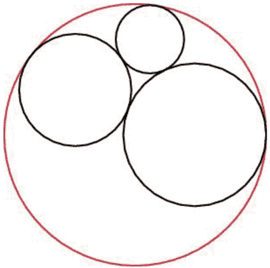
We can only have at most one "inverted" circle!

Descartes Quadruples

Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.





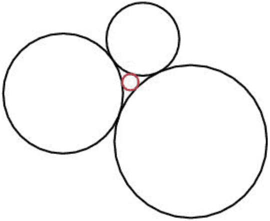
We can only have at most one "inverted" circle!

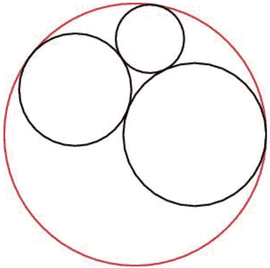
Theorem of Apollonius

Descartes Quadruples

Definition

Descartes quadruple: four mutually tangent circles with disjoint interiors.





We can only have at most one "inverted" circle!

Theorem of Apollonius

If three circles are mutually tangent, there are two other circles that are tangent to all three.

The Descartes Equation

The Descartes Equation

Definition

The Descartes Equation

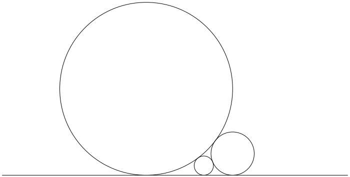
Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

The Descartes Equation

Definition

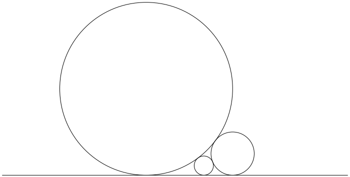
The *curvature* of a circle with radius r is defined to be $1/r$.



The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.

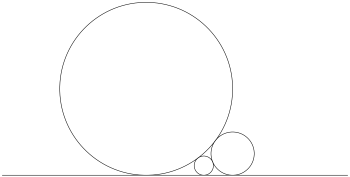


Circle with infinite radius!

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



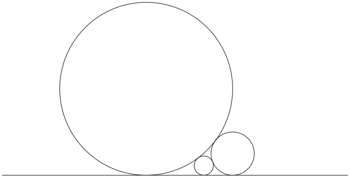
Circle with infinite radius!

Descartes Equation

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius!

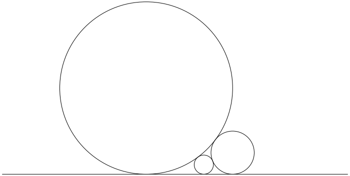
Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

The Descartes Equation

Definition

The *curvature* of a circle with radius r is defined to be $1/r$.



Circle with infinite radius!

Descartes Equation

If four mutually tangent circles have curvatures a, b, c, d then

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2).$$

The Descartes Equation

The Descartes Equation

Corollary

The Descartes Equation

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d' have curvatures

The Descartes Equation

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d′ have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

The Descartes Equation

Corollary

If three mutually tangent circles have curvatures a, b, and c, then the two circles of Apollonius, d and d′ have curvatures

$$d = a + b + c + 2\sqrt{ab + ac + bc}$$
$$d' = a + b + c - 2\sqrt{ab + ac + bc}$$

Moreover, $d + d' = 2(a + b + c)$.

Apollonian Circle Packings

Apollonian Circle Packings

The Key Relation

Apollonian Circle Packings

The Key Relation

$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

Apollonian Circle Packings

The Key Relation

$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

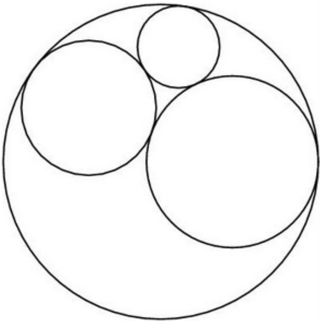
If a, b, c, d are integers, then d' is an integer!

Apollonian Circle Packings

The Key Relation

$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

If a, b, c, d are integers, then d' is an integer!

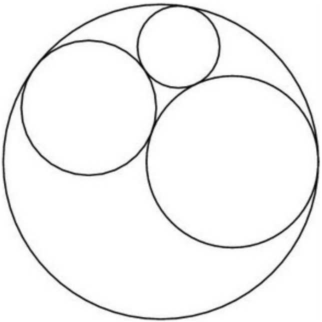


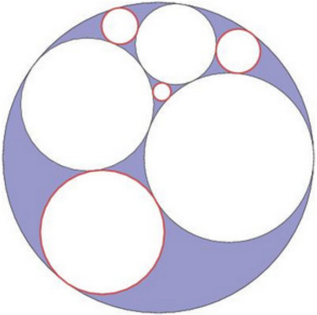
Apollonian Circle Packings

The Key Relation

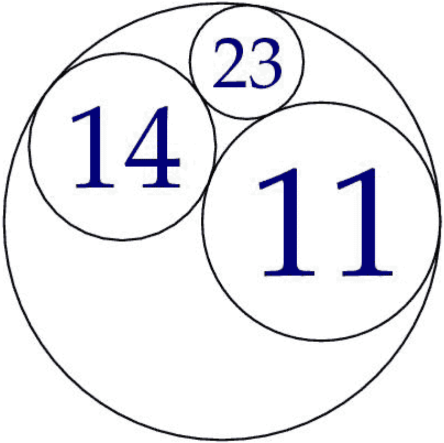
$$d + d' = 2(a + b + c) \implies d' = 2(a + b + c) - d$$

If a, b, c, d are integers, then d' is an integer!



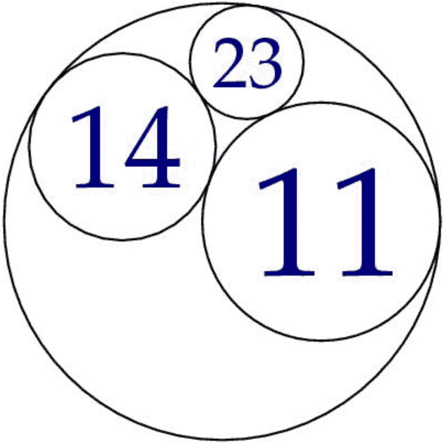


Apollonian Circle Packings



[1] Images from: AMS "When Kissing Involves Trigonometry"

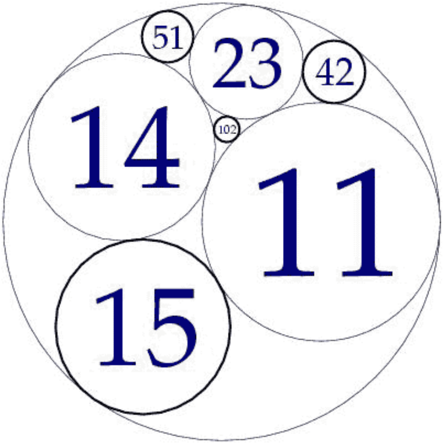
Apollonian Circle Packings



$[-6, 11, 14, 23]$ [1]

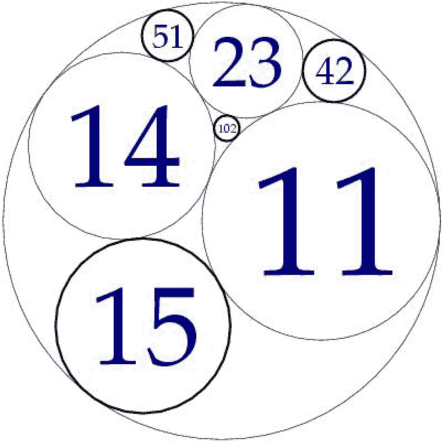
[1] Images from: AMS "When Kissing Involves Trigonometry"

Apollonian Circle Packings



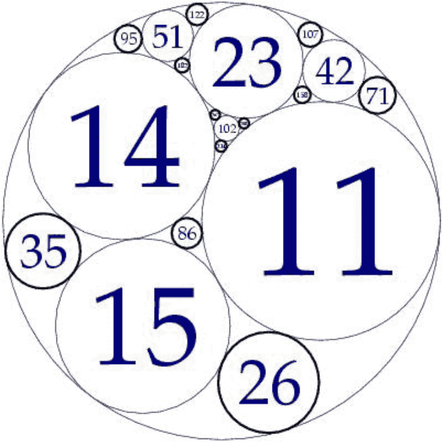
$[-6, 11, 14, 23]$

Apollonian Circle Packings

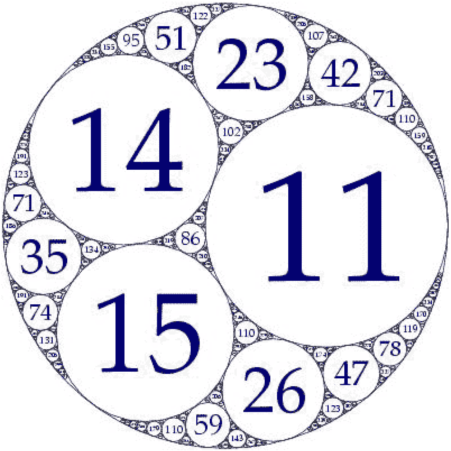


$[-6, 11, 14, 23]$ reduces to $[-6, 11, 14, 15]$

Apollonian Circle Packings



$[-6, 11, 14, 15]$



11
14
15
23
26
35
42
47
51
59
71
71
74
78
86
95
102
107
110
110
110
119
122
123
123
131
134
143

$[-6, 11, 14, 15]$

Quadratic Forms

Quadratic Forms

Definition

Quadratic Forms

Definition

Binary quadratic form (BQF)

$$Q(x, y) = Ax^2 + Bxy + Cy^2 = [A, B, C]$$

with $A, B, C \in \mathbb{Z}$

Quadratic Forms

Definition

Binary quadratic form (BQF)

$$Q(x, y) = Ax^2 + Bxy + Cy^2 = [A, B, C]$$

with $A, B, C \in \mathbb{Z}$

Fixing a circle a, the values of $f_a(x, y) - a$ with $\gcd(x, y) = 1$,

Quadratic Forms

Definition

Binary quadratic form (BQF)

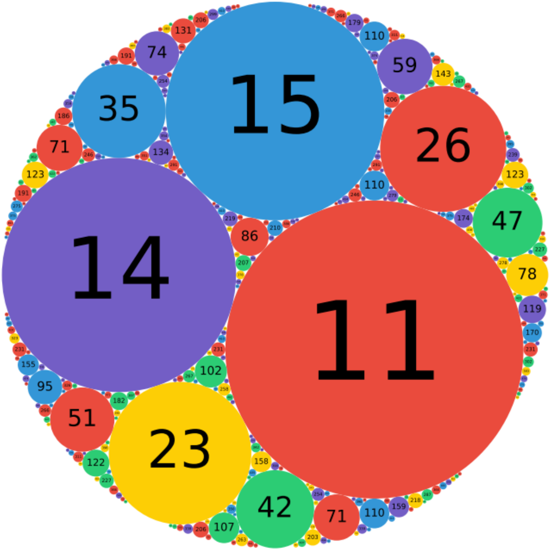
$$Q(x, y) = Ax^2 + Bxy + Cy^2 = [A, B, C]$$

with $A, B, C \in \mathbb{Z}$

Fixing a circle a, the values of $f_a(x, y) - a$ with $\gcd(x, y) = 1$, a primitive integral binary quadratic form, are curvatures of circles tangent to a (Sarnak, Graham-Lagarias-Mallows-Wilks-Yan)

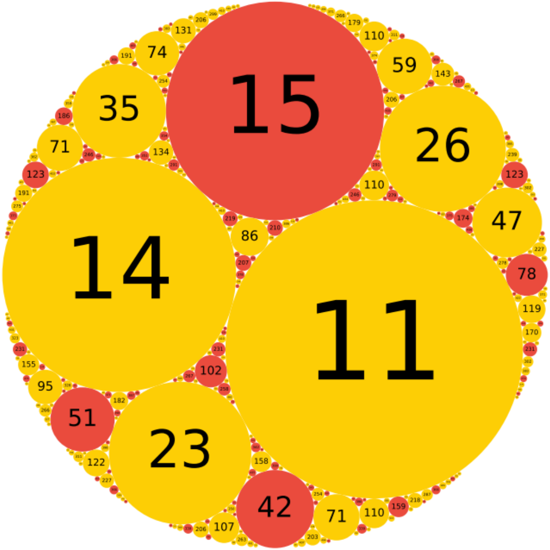
Curvatures Mod 5

Curvatures Mod 5



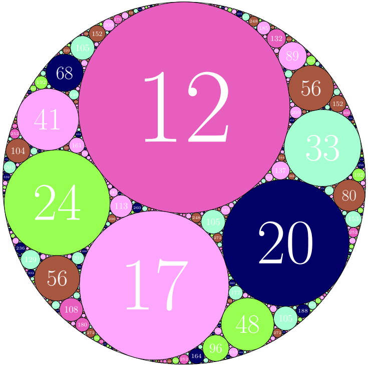
Curvatures Mod 3

Curvatures Mod 3

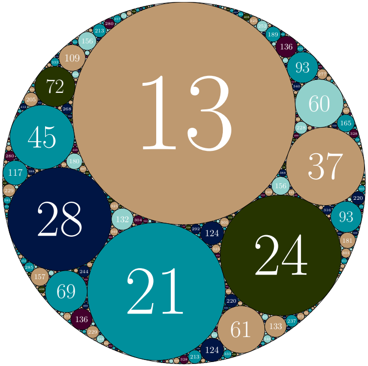


Admissible Residues

Admissible Residues



$[-7, 12, 17, 20]$



$[-8, 13, 21, 24]$

Admissible Residues

Admissible Residues

residues mod 24
0,1,4,9,12,16
0,5,8,12,20,21
0,4,12,13,16,21
0,8,9,12,17,20
3,6,7,10,15,18,19,22
2,3,6,11,14,15,18,23

Admissible Residues

Type	residues mod 24
(6,1)	0,1,4,9,12,16
(6,5)	0,5,8,12,20,21
(6,13)	0,4,12,13,16,21
(6,17)	0,8,9,12,17,20
(8,7)	3,6,7,10,15,18,19,22
(8,11)	2,3,6,11,14,15,18,23

Admissible Residues

Type	residues mod 24
(6,1)	0,1,4,9,12,16
(6,5)	0,5,8,12,20,21
(6,13)	0,4,12,13,16,21
(6,17)	0,8,9,12,17,20
(8,7)	3,6,7,10,15,18,19,22
(8,11)	2,3,6,11,14,15,18,23

Theorem (Fuchs)

If a congruence obstruction appears, then it appears modulo 24.

Local-to-global

Local-to-global

The Local-Global Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan '03, Fuchs-Sanden '11)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

Local-to-global

The Local-Global Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan '03, Fuchs-Sanden '11)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

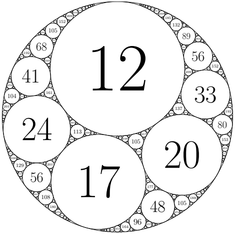
Body of work by Graham-Lagarias-Mallows-Wilks-Yan, Sarnak, Bourgain-Fuchs, Bourgain-Kontorovich, Fuchs-Stange-Zhang

Local-to-global

The Local-Global Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan '03, Fuchs-Sanden '11)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

Body of work by Graham-Lagarias-Mallows-Wilks-Yan, Sarnak, Bourgain-Fuchs, Bourgain-Kontorovich, Fuchs-Stange-Zhang



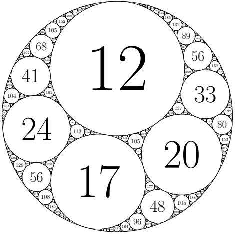
$[-7, 12, 17, 20]$ type $(6, 17)$

Local-to-global

The Local-Global Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan '03, Fuchs-Sanden '11)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

Body of work by Graham-Lagarias-Mallows-Wilks-Yan, Sarnak, Bourgain-Fuchs, Bourgain-Kontorovich, Fuchs-Stange-Zhang



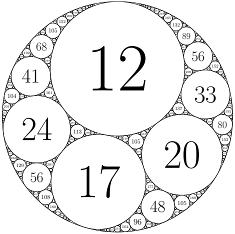
$[-7, 12, 17, 20]$ type $(6, 17) \implies 0, 8, 9, 12, 17, 20$

Local-to-global

The Local-Global Conjecture. (Graham-Lagarias-Mallows-Wilks-Yan '03, Fuchs-Sanden '11)

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, and all sufficiently large integers satisfying this condition appear.

Body of work by Graham-Lagarias-Mallows-Wilks-Yan, Sarnak, Bourgain-Fuchs, Bourgain-Kontorovich, Fuchs-Stange-Zhang



$[-7, 12, 17, 20]$ type $(6, 17) \implies 0, 8, 9, 12, 17, 20$: no room for 8,9,32,...

History

$$\mathcal{K}(N) := \{n \leq N : n \text{ is a curvature}\}$$

History

$$\mathcal{K}(N) := \{n \leq N : n \text{ is a curvature}\}$$

- Number of circles of curvatures less than T grows like $T^{\alpha+o(1)}$ with $\alpha = 1.30... =$ Hausdorff dim (Boyd, McMullen, Kontorovich-Oh)

History

$$\mathcal{K}(N) := \{n \leq N : n \text{ is a curvature}\}$$

- Number of circles of curvatures less than T grows like $T^{\alpha+o(1)}$ with $\alpha = 1.30... =$ Hausdorff dim (Boyd, McMullen, Kontorovich-Oh)
- $\mathcal{K}(N) \gg N$ (Graham-Lagarias-Mallows-Wilks-Yan)

History

$$\mathcal{K}(N) := \{n \le N : n \text{ is a curvature}\}$$

- Number of circles of curvatures less than T grows like $T^{\alpha+o(1)}$ with $\alpha = 1.30... =$ Hausdorff dim (Boyd, McMullen, Kontorovich-Oh)
- $\mathcal{K}(N) \gg N$ (Graham-Lagarias-Mallows-Wilks-Yan)
- $\mathcal{K}(N) \gg \frac{N}{\sqrt{\log(N)}}$ (Sarnak)

History

$$\mathcal{K}(N) := \{n \le N : n \text{ is a curvature}\}$$

- Number of circles of curvatures less than T grows like $T^{\alpha+o(1)}$ with $\alpha = 1.30... =$ Hausdorff dim (Boyd, McMullen, Kontorovich-Oh)
- $\mathcal{K}(N) \gg N$ (Graham-Lagarias-Mallows-Wilks-Yan)
- $\mathcal{K}(N) \gg \frac{N}{\sqrt{\log(N)}}$ (Sarnak)
- $\mathcal{K}(N) \gg N$ (positive density) (Bourgain-Fuchs)

History

$$\mathcal{K}(N) := \{n \leq N : n \text{ is a curvature}\}$$

- Number of circles of curvatures less than T grows like $T^{\alpha+o(1)}$ with $\alpha = 1.30... =$ Hausdorff dim (Boyd, McMullen, Kontorovich-Oh)
- $\mathcal{K}(N) \gg N$ (Graham-Lagarias-Mallows-Wilks-Yan)
- $\mathcal{K}(N) \gg \frac{N}{\sqrt{\log(N)}}$ (Sarnak)
- $\mathcal{K}(N) \gg N$ (positive density) (Bourgain-Fuchs)
- $\exists \eta > 0$, $\mathcal{K}(N) = kN + O(N^{1-\eta})$ (density 1) (Bourgain-Kontorovich)
- $\exists \eta > 0$, $\mathcal{K}(N) = kN + O(N^{1-\eta})$ for a larger class of packings (Fuchs-Stange-Zhang)

Computational Evidence

Computational Evidence

Curvature c is *missing* in ACP if curvatures $\equiv c \pmod{24}$ appear in ACP but c does not

Computational Evidence

Curvature c is *missing* in ACP if curvatures $\equiv c \pmod{24}$ appear in ACP but c does not

Fuchs-Sanden computed curvatures up to:

Computational Evidence

Curvature c is *missing* in ACP if curvatures $\equiv c \pmod{24}$ appear in ACP but c does not

Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } [-1, 2, 2, 3]$$

Computational Evidence

Curvature c is *missing* in ACP if curvatures $\equiv c \pmod{24}$ appear in ACP but c does not

Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } [-1, 2, 2, 3]$$

$$5 \cdot 10^8 \text{ for } [-11, 21, 24, 28]$$

Computational Evidence

Curvature c is *missing* in ACP if curvatures $\equiv c \pmod{24}$ appear in ACP but c does not

Fuchs-Sanden computed curvatures up to:

$$10^8 \text{ for } [-1, 2, 2, 3]$$

$$5 \cdot 10^8 \text{ for } [-11, 21, 24, 28]$$

For $[-11, 21, 24, 28]$, there were still a small number (up to 0.013%) of missing curvatures in the range $(4 \cdot 10^8, 5 \cdot 10^8)$ for residue classes $0, 4, 12, 16 \pmod{24}$

Summer 2023 REU

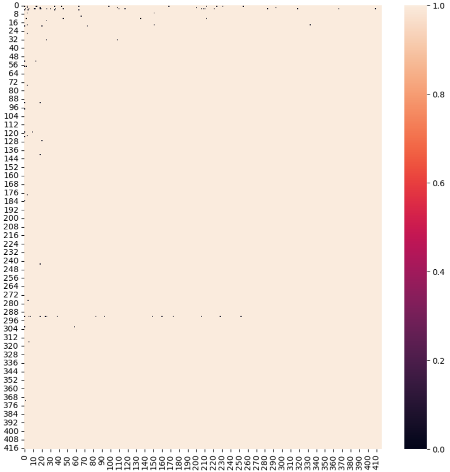
- Fix a pair of curvatures and see what packings contain them

- Fix a pair of curvatures and see what packings contain them
- For an acceptable pair, black dot if no packing contains them

Summer 2023 REU

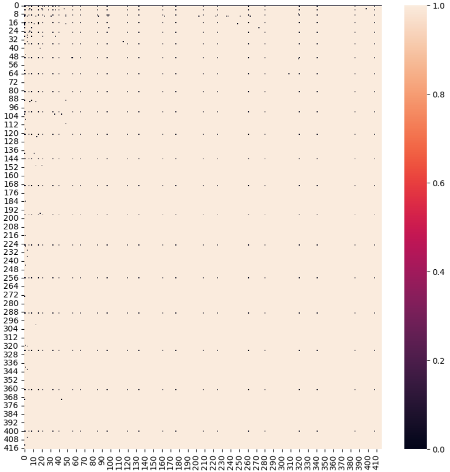
- Fix a pair of curvatures and see what packings contain them
- For an acceptable pair, black dot if no packing contains them
- Local-to-global: fintely many black dots for a row or column

Usual Graph



Residue classes: 12 (mod 24) and 13 (mod 24)

Weird Graph



Residue classes: 0 (mod 24) and 8 (mod 24)

Local-to-global conjecture is false

(H.-K.-Rickards-Stange)

The Apollonian circle packing generated by $[-3, 5, 8, 8]$ has no square curvatures.

A circle has a 'residuosity'

A circle has a 'residuosity'

1. Fix circle C with curvature n

A circle has a 'residuosity'

1. Fix circle C with curvature n
2. Tangent curvatures are $f_C(x, y) - n$

A circle has a 'residuosity'

1. Fix circle C with curvature n
2. Tangent curvatures are $f_C(x, y) - n$
3. Modulo n and equivalences, values are Ax^2

A circle has a 'residuosity'

1. Fix circle C with curvature n
2. Tangent curvatures are $f_C(x, y) - n$
3. Modulo n and equivalences, values are Ax^2
4. Kronecker symbol $\left(\frac{Ax^2}{n}\right)$ cannot take both values 1 and -1

A circle has a 'residuosity'

1. Fix circle C with curvature n
2. Tangent curvatures are $f_C(x, y) - n$
3. Modulo n and equivalences, values are Ax^2
4. Kronecker symbol $\left(\frac{Ax^2}{n}\right)$ cannot take both values 1 and -1
5. Define $\chi_2(C) = 1$ or -1 according to above

A circle has a 'residuosity'

1. Fix circle C with curvature n
2. Tangent curvatures are $f_C(x, y) - n$
3. Modulo n and equivalences, values are Ax^2
4. Kronecker symbol $\left(\frac{Ax^2}{n}\right)$ cannot take both values 1 and -1
5. Define $\chi_2(C) = 1$ or -1 according to above
6. Note: $\chi_2(C) = \left(\frac{a}{n}\right)$ for curvature a coprime and tangent to C

A circle has a 'residuosity'

1. Fix circle C with curvature n
2. Tangent curvatures are $f_C(x, y) - n$
3. Modulo n and equivalences, values are Ax^2
4. Kronecker symbol $\left(\frac{Ax^2}{n}\right)$ cannot take both values 1 and -1
5. Define $\chi_2(C) = 1$ or -1 according to above
6. Note: $\chi_2(C) = \left(\frac{a}{n}\right)$ for curvature a coprime and tangent to C

A packing has a 'residuosity'

1. Take C_1, C_2 tangent circles in ACP with curvatures a, b respectively that are coprime

A packing has a 'residuosity'

1. Take C_1, C_2 tangent circles in ACP with curvatures a, b respectively that are coprime
2. Quadratic Reciprocity!!

A packing has a 'residuosity'

1. Take C_1, C_2 tangent circles in ACP with curvatures a, b respectively that are coprime
2. Quadratic Reciprocity!!

$$\chi_2(C_1)\chi_2(C_2) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1 \implies \chi_2(C_1) = \chi_2(C_2)$$

A packing has a 'residuosity'

1. Take C_1, C_2 tangent circles in ACP with curvatures a, b respectively that are coprime
2. Quadratic Reciprocity!!

$$\chi_2(C_1)\chi_2(C_2) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1 \implies \chi_2(C_1) = \chi_2(C_2)$$

3. Any two circles in ACP are connected by a path of pairwise coprime curvatures

A packing has a 'residuosity'

1. Take C_1, C_2 tangent circles in ACP with curvatures a, b respectively that are coprime
2. Quadratic Reciprocity!!

$$\chi_2(C_1)\chi_2(C_2) = \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1 \implies \chi_2(C_1) = \chi_2(C_2)$$

3. Any two circles in ACP are connected by a path of pairwise coprime curvatures
4. $\chi_2(C)$ is independent of choice of circle C!!

No Squares in $[-3, 5, 8, 8]$

No Squares in $[-3, 5, 8, 8]$

1. Let $\mathcal{A}$ be the ACP generated by $[-3, 5, 8, 8]$, compute

$$\chi_2(\mathcal{A}) = \left(\frac{8}{5}\right) = \left(\frac{3}{5}\right) = -1$$

No Squares in $[-3, 5, 8, 8]$

1. Let $\mathcal{A}$ be the ACP generated by $[-3, 5, 8, 8]$, compute

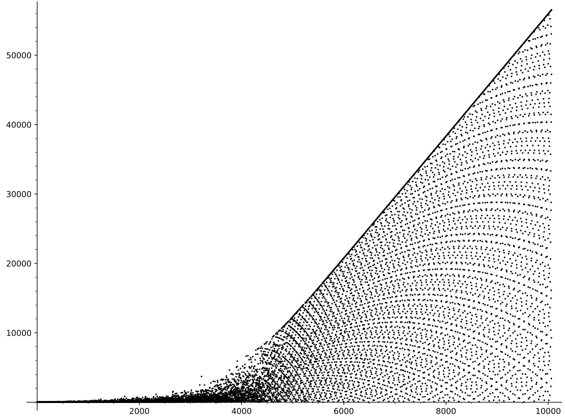
$$\chi_2(\mathcal{A}) = \left(\frac{8}{5}\right) = \left(\frac{3}{5}\right) = -1$$

2. No circle can be tangent to a square

The New Conjecture

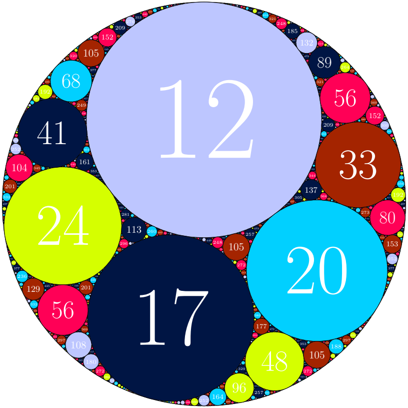
Type	Quadratic	Quartic	L-G false	L-G open
$(6, 1, 1, 1)$				$0, 1, 4, 9, 12, 16$
$(6, 1, 1, -1)$		$n^4, 4n^4, 9n^4, 36n^4$	$0, 1, 4, 9, 12, 16$	
$(6, 1, -1)$	$n^2, 2n^2, 3n^2, 6n^2$		$0, 1, 4, 9, 12, 16$	
$(6, 5, 1)$	$2n^2, 3n^2$		$0, 8, 12$	$5, 20, 21$
$(6, 5, -1)$	$n^2, 6n^2$		$0, 12$	$5, 8, 20, 21$
$(6, 13, 1)$	$2n^2, 6n^2$		0	$4, 12, 13, 16, 21$
$(6, 13, -1)$	$n^2, 3n^2$		$0, 4, 12, 16$	$13, 21$
$(6, 17, 1, 1)$	$3n^2, 6n^2$	$9n^4, 36n^4$	$0, 9, 12$	$8, 17, 20$
$(6, 17, 1, -1)$	$3n^2, 6n^2$	$n^4, 4n^4$	$0, 9, 12$	$8, 17, 20$
$(6, 17, -1)$	$n^2, 2n^2$		$0, 8, 9, 12$	$17, 20$
$(8, 7, 1)$	$3n^2, 6n^2$		$3, 6$	$7, 10, 15, 18, 19, 22$
$(8, 7, -1)$	$2n^2$		18	$3, 6, 7, 10, 15, 19, 22$
$(8, 11, 1)$				$2, 3, 6, 11, 14, 15, 17, 23$
$(8, 11, -1)$	$2n^2, 3n^2, 6n^2$		$2, 3, 6, 18$	$11, 14, 15, 23$

differences between successive missing curvatures



Successive differences of missing curvatures in the packing $(-4, 5, 20, 21)$. The quadratic families $2n^2$ and $3n^2$ begin to predominate (the sporadic set has 3659 elements $< 10^{10}$, and occur increasingly sparsely.)

Thank You!!


12
17
20
24
33
41
56
68
89
105
104
113
132
137
152
161
164
177
185
188
201
209
48
96
80
153
129
56
108
180